# MATHEMATICS METHODS

# MAWA Semester 1 (Unit 3) Examination 2018
# Calculator-assumed

# Marking Key

## Section Two: Calculator-assumed (100 Marks)

### Question 9 (5 marks)

**Solution**

$(0,-6) \Rightarrow d=-6$

$f(x)=ax^4+bx^2+cx+d$

$f'(x)=4ax^3+2bx+c$

$f'(0)=0$

$0=4a(0)^3+2b(0)+c$

$\therefore c=0$

$f'(x)=4ax^3+2bx$

$f'(1)=0$

$0=4a(1)^3+2b(1)$

$0=4a+2b$

$b=-2a$ ①

$(1,-8) \Rightarrow -8=a+b-6$ ②

ie $-8=a-2a-6$

ie $a=2$

$\therefore b=-4$

$\therefore y=2x^4-4x^2-6$

| Mathematical behaviours | Marks |
|---|---|
| • uses (0,- 6) to determine $d$ | 1 |
| • differentiates $f(x)$ and uses $f'(0)=0$ to obtain $c$ | 1 |
| • states $f'(1)=0$ and states relationship between $a$ and $b$ | 1 |
| • uses (1,-8) to determine relationship between $a$ and $b$ | 1 |
| • solves simultaneous equations to determine $a$ and $b$ | 1 |

**Question 10(a)** **(4 marks)**

| Solution | |
|---|---|
| Let $F$ denote the number of questions that Fiona answers correctly, assuming that she is guessing. Then $P(F \geq 3)$ is probability that Fiona passes.<br>Similarly, if $G$ denotes the number of questions that Gary answers correctly, assuming that he is guessing, then $P(G \geq 10)$ is probability that Gary passes.<br>Now $F \sim B(6,1/5)$, and $G \sim B(20,1/3)$.<br>$P(F \geq 3) \approx 0.0989$ and $P(G \geq 10) \approx 0.0919$<br>Since the probability that Gary passes via guessing is less than the probability that Fiona passes via guessing, we can say that Gary is luckier. | |
| Mathematical behaviours | Marks |
| • recognizes the binomial probabilities<br>• evaluates probabilities<br>• justifies who is luckier | 1<br>1+1<br>1 |

**Question 10 (b) (i)** **(2 marks)**

| Solution | |
|---|---|
| Let $L$ denote the number of light bulbs that fail in a random sample of 100. Then $L \sim B(100,0.04)$, if the manufacturer is correct.<br>Then $P(L \geq 15) = 1.082 \times 10^{-5}$ $(very\ very\ small)$ | |
| Mathematical behaviours | Marks |
| • recognizes binomial probability with correct parameters<br>• states probability | 1<br>1 |

**Question 10 (b) (ii)** **(2 marks)**

| Solution | |
|---|---|
| Because the probability that such a large number of bulbs fail if the manufacturer's claim is correct, is very, very small, there is strong reason to doubt the validity of the claim. | |
| Mathematical behaviours | Marks |
| • correct conclusion<br>• justifies reasoning | 1<br>1 |

**Question 11** **(3 marks)**

| Solution | |
|---|---|
| $\frac{dI}{dt} = 0.03I$<br>$\therefore I = I_0 e^{0.03t}$<br>For the number of infected fruit to double,<br>$2I_0 = I_0 e^{0.03t}$<br>ie $t \approx 23.1$days | |
| Mathematical behaviours | Marks |
| • recognises and states exponential growth formula for $I$<br>• uses relationship $I = 2I_0$<br>• states solution | 1<br>1<br>1 |

**Question 12 (a)** **(2 marks)**

| Solution | |
|---|---|
| $X_{max} \leq n$ if and only if the number of each die is no more than $n$<br>For each die this occurs with probability $\frac{n}{6}$<br>Since the dice are independent, the probability that this occurs both dice is $\left(\frac{n}{6}\right)^2$ | |
| Mathematical behaviours | Marks |
| • observes that both numbers must be at most $n$<br>• uses independence to justify multiplicative formula | 1<br>1 |

**Question 12 (b)** **(2 marks)**

| Solution | |
|---|---|
| From part (a) $P(X_{max} \leq 4) = 4/9$<br>So Vanessa's expected winnings from each \$1 she bets is $\$\left(\frac{4}{9} - \frac{5}{9}\right) = -\$\frac{1}{9}$<br>So her expected return from 100 \$1 bets is a loss of $\$\frac{100}{9} \cong \$11.11$ | |
| Mathematical behaviours | Marks |
| • correct expected value for a \$1 bet.<br>• correct final answer | 1<br>1 |

**Question 12 (c)** **(2 marks)**

Solution

| $n$ | $P(X_{max} = n)$ |
|---|---|
| 1 | 1/36 |
| 2 | 4/36 – 1/36 = 3/36 |
| 3 | 9/36 – 4/36 = 5/36 |
| 4 | 16/36 – 9/36 = 7/36 |
| 5 | 25/36 – 16/36 = 9/36 |
| 6 | 36/36 – 25/36 = 11/36 |

| Mathematical behaviours | Marks |
|---|---|
| • uses subtraction to obtain individual probabilities from the cumulative ones in part (a)<br>• correct answers in all 5 outstanding cases | 1<br>1 |

**Question 12 (d)** **(2 marks)**

Solution

Directly from calculator, or via:

| $n$ | $P(X_{max} = n)$ | $n \times P(X_{max} = n)$ |
|---|---|---|
| 1 | 1/36 | 1/36 |
| 2 | 3/36 | 6/36 |
| 3 | 5/36 | 15/36 |
| 4 | 7/36 | 28/36 |
| 5 | 9/36 | 45/36 |
| 6 | 11/36 | 66/36 |

So $E(X_{max}) = \Sigma(n \times P(X_{max} = n)) = \dfrac{161}{36} \approx 4.47$

| Mathematical behaviours | Marks |
|---|---|
| • writes a calculation for expected value<br>• determines expected value | 1<br>1 |

**Question 12 (e)** **(2 marks)**

Solution

Directly from calculator, or via

| $n$ | $P(X_{max} = n)$ | $n^2 \times P(X_{max} = n)$ |
|---|---|---|
| 1 | 1/36 | 1/36 |
| 2 | 3/36 | 12/36 |
| 3 | 5/36 | 45/36 |
| 4 | 7/36 | 112/36 |
| 5 | 9/36 | 225/36 |
| 6 | 11/36 | 396/36 |

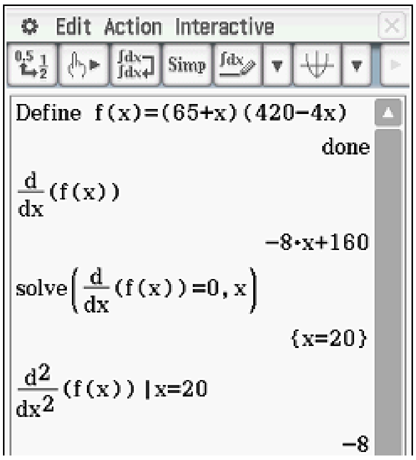
So $Var(E_{max}) = E\left(X_{max}{}^2\right) - E(X_{max})^2 = \dfrac{7911}{36} - \left(\dfrac{161}{36}\right)^2 = \dfrac{2555}{1296} \approx 1.97$

| Mathematical behaviours | Marks |
|---|---|
| • calculates $E(X^2)$ correctly<br>• calculates variance correctly | 1<br>1 |

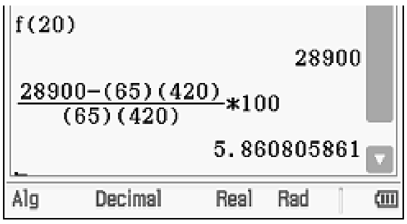
**Question 12 (f)** **(3 marks)**

| Solution | |
|---|---|
| If a large number of dice are thrown, *m*, say, then<br>$P(Y_{max} \leq 5) = \left(\frac{5}{6}\right)^m \approx 0.$<br>So $Y_{max}$ is almost certainly equal to 6<br>So $E(Y_{max}) \approx 6$ and $Var(Y_{max}) \approx 0$ | |
| **Mathematical behaviours** | **Marks** |
| • states that $Y_{max}$ is almost certainly equal to 6<br>• correct answer for $E(Y_{max})$<br>• correct answer for $Var(Y_{max})$ | 1<br>1<br>1 |

**Question 13 (a)** **(4 marks)**

| Solution | |
|---|---|
| Let $y$ represent the yield from the trees<br>Let $x$ represent the number of extra trees planted<br><br>$y = (65 + x)(420 - 4x)$<br>$y = 27300 + 160x - 4x^2$<br>$y' = 160 - 8x$<br>$0 = 160 - 8x$<br>$x = 20$<br>$y'' = -8 \Rightarrow x = 20$ is max<br>Macintosh orchard should plant an additional 20 trees for optimal yield. |  |
| **Mathematical behaviours** | **Marks** |
| • clearly identifies variables used in equation<br>• correct 'yield' equation<br>• differentiates and solves $y' = 0$<br>• justifies that maximum is found and states solution | 1<br>1<br>1<br>1 |

**Question 13 (b)** **(1 mark)**

| Solution | |
|---|---|
| Current Yield: 27 300 oranges<br>Optimal Yield: 28 900 oranges<br><br>$\frac{28900 - 27300}{27300} \times 100\%$<br>$\approx 5.86\%$<br><br>Therefore there would be a 5.9% increase in yield after planting 10 additional trees. |  |
| **Mathematical behaviours** | **Marks** |
| • states correct answer | 1 |

**Question 14 (a)** **(3 marks)**

Solution

upper limit $= 2\times(f(0)+f(2)+f(4))$

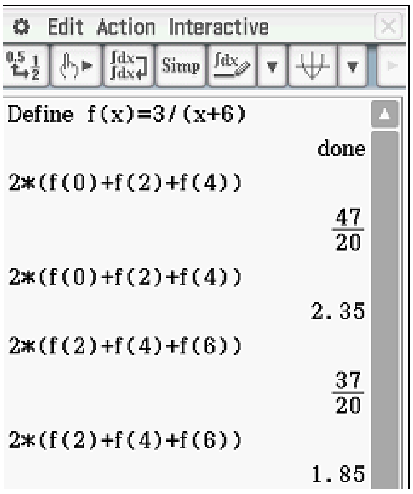
$= 2\times(\frac{3}{6}+\frac{3}{8}+\frac{3}{10})$

$= \frac{47}{20}$

lower limit $= 2\times(f(2)+f(4)+f(6))$

$= 2\times(\frac{3}{8}+\frac{3}{10}+\frac{3}{12})$

$= \frac{37}{20}$




$\int_0^6 f(x)\,dx$ represents the area under the curve from $x$ = 0 to $x$= 6, bounded by the $x$ axis.

The exact area will lie between the upper limit and lower limit.

| Mathematical behaviours | Marks |
|---|---|
| • shows a calculation to determine an upper limit | 1 |
| • shows a calculation to determine a lower limit | 1 |
| • explains the limits in terms of area | 1 |

**Question 14 (b)** **(1 mark)**

Solution

Using more rectangles would enable the rectangles to more closely approximate the shape of the function. Hence the error involved in approximating $\int_0^6 f(x)\,dx$ is less and the interval obtained will decrease.

| Mathematical behaviours | Marks |
|---|---|
| • explains why the interval will decrease | 1 |

**Question 14 (c)** **(1 mark)**

Solution

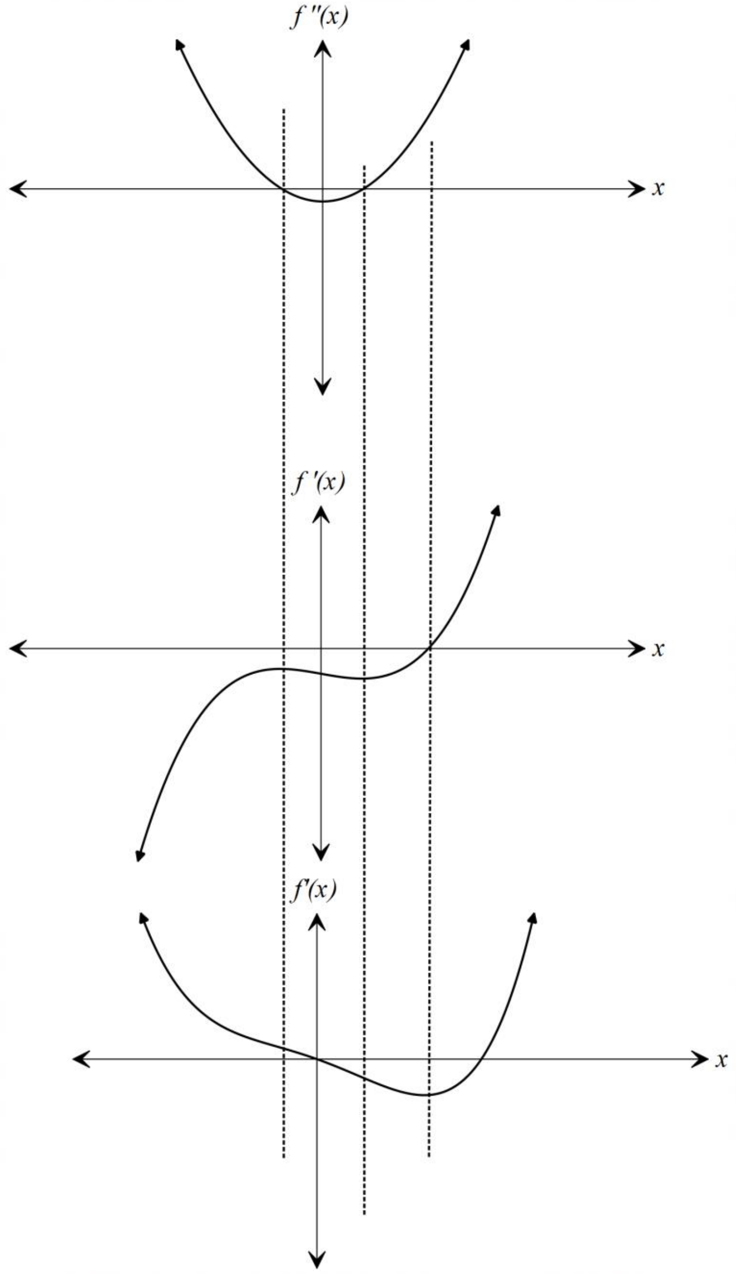
$$\int_0^6 \frac{3}{x+6}\,dx = 2.079441542$$

$\approx 2.079$

| Mathematical behaviours | Marks |
|---|---|
| • states correct answer to 4 significant figures | 1 |

**Question 15** **(6 marks)**

Solution




| Mathematical behaviours | Marks |
|---|---|
| For $f'(x)$ | |
| • point of inflection is on $y$ axis | 1 |
| • maxima/minima align with roots of $f''(x)$ | 1 |
| • correct shape | 1 |
| For $f(x)$ | |
| • stationary points align with $x$-intercepts from $f'(x)$ | 1 |
| • points of inflection align with the roots of $f''(x)$ | 1 |
| • correct shape | 1 |

**Question 16 (a)** **(2 marks)**

**Solution**

$s'(t) = 6t^2 - 38t + 52$

$s'(5) = 6(5)^2 - 38(5) + 52$

$s'(5) = 12$

The rate of change of displacement with respect to time at 5 seconds is 12 m/s.

Define $f(x)=2\cdot x^3-19\cdot x^2+52\cdot x$ done

$\frac{d}{dx}(f(x))|x=5$ 12

| Mathematical behaviours | Marks |
|---|---|
| • determines $s'(t)$ | 1 |
| • determines the rate of change with units | 1 |

**Question 16 (b)** **(3 marks)**

**Solution**

$v(t) = 6t^2 - 38t + 52$

$v(0) = 52$

$v(t) = 0 \Rightarrow 6t^2 - 38t + 52 = 0$

ie $v = 2, \frac{13}{3}$

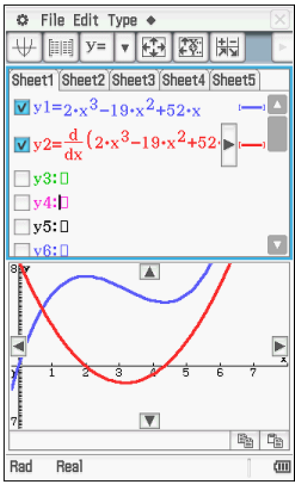
In graph mode $s(t) > 0 \quad \forall t > 0,$

$v(t) < 0$ for $2 < t < \frac{13}{3},$

Hence the particle is moving towards the origin for $2 < t < \frac{13}{3}$

$\text{solve}\left(\frac{d}{dx}(f(x))=0, x\right)$

$\left\{x=2, x=\frac{13}{3}\right\}$




| Mathematical behaviours | Marks |
|---|---|
| • solves $v(t) = 0$ | 1 |
| • demonstrates $s(t) > 0 \quad \forall t$ | 1 |
| • states interval with correct symbols | 1 |

**Question 17 (a)** **(3 marks)**

| Solution |
|---|
| The required probabilities are the ratios of the numbers of favourable choices to the number of all possible choices.<br>The number of all possible choices is $\binom{5}{2} = 10$ |

| $x$ | $P(A = x)$ |
|---|---|
| 0 | $\binom{2}{2} \div 10 = 1/10$ |
| 1 | $\binom{2}{1}\binom{3}{1} \div 10 = 6/10$ |
| 2 | $\binom{3}{2} \div 10 = 3/10$ |

| Mathematical behaviours | Marks |
|---|---|
| • uses combinations to determine numerators<br>• uses combinations to determine denominators<br>• evaluates all probabilities | 1<br>1<br>1 |

**Question 17 (b)** **(3 marks)**

| Solution |
|---|
| $E(A) = 0 \times 0.1 + 1 \times 0.6 + 2 \times 0.3 = 1.2$<br>$E(A^2) = 0 \times 0.1 + 1 \times 0.6 + 4 \times 0.3 = 1.8$<br>So $Var(A) = E(A^2) - E(A)^2 = 1.8 - 1.44 = 0.36$<br>In summary, the expected value of $A$ is 1.2 and the variance is 0.36 |

| Mathematical behaviours | Marks |
|---|---|
| • evaluates $E(A)$<br>• evaluates $E(A^2)$<br>• evaluates $Var(A)$ | 1<br>1<br>1 |

**Question 17 (c)** **(3 marks)**

| Solution |
|---|
| $B$ has a binomial distribution because it represents the sum of two independent trials (choosing mugs) with the same probability of 'success' in each trial<br>$A$ does not have a binomial distribution because the trials are not independent, i.e. the outcome of the first trial affects the probabilities in the second trial |

| Mathematical behaviours | Marks |
|---|---|
| • independence of trials noted (for $B$)<br>• unchanged probabilities noted (for $B$)<br>• probabilities for the second choice affected by the outcome of the first choice (for $A$) | 1<br>1<br>1 |

**Question 17 (d)** **(5 marks)**

| Solution |
|---|
| The correct answer is the expected value of $C$.<br>$C = x$ if and only if the $x^{th}$ mug chosen is unchipped and exactly 1 of the previous $n-1$ chosen mugs is unchipped. So |

| $x$ | $P(C = x)$ |
|---|---|
| 2 | $\frac{3}{5} \times \frac{2}{4} = 3/10$ |
| 3 | $\frac{6}{10} \times \frac{2}{3} = 4/10$ |
| 4 | $\frac{3}{10} \times \frac{2}{2} = 3/10$ |

So $E(C) = 2 \times 0.3 + 3 \times 0.4 + 4 \times 0.3 = 3$
So Caitlin can expect to choose, on average, 3 mugs.

| Mathematical behaviours | Marks |
|---|---|
| • recognizes $E(C)$ as the correct answer | 1 |
| • evaluates individual probabilities | 1+1+1 |
| • evaluates $E(C)$ | 1 |

**Question 18 (a)** **(1 mark)**

| Solution |
|---|
| $F(x) = \int_{-4}^{x} f(t)\, dt$ for $-4 \leq x \leq 1$.<br>$F(-4) = \int_{-4}^{-4} f(t)\, dt = 0$ |

| Mathematical behaviours | Marks |
|---|---|
| • determines $F(-4)$ | 1 |

**Question 18 (b)** **(2 marks)**

| Solution |
|---|
| $F'(x) = \frac{d}{dx}\int_{-4}^{x} f(t)\, dt$ .<br>$= f(x)$<br>Stationary points occur at $F'(x) = f(x) = 0$, hence at $x = -4,-2$ and $0$. |

| Mathematical behaviours | Marks |
|---|---|
| • recognizes and applies the fundamental theorem | 1 |
| • identifies three stationary points | 1 |

**Question 18 (c)** **(2 marks)**

| Solution | |
|---|---|
| $F$ is increasing where $F'(x) > 0$, hence where $f(x)$ is greater than $0$<br>$\therefore F$ is increasing for $-4 < x < -2$ and $0 < x \le 1$. | |
| **Mathematical behaviours** | **Marks** |
| • identifies one interval for which $F$ is increasing | 1 |
| • states, with correct symbols, both intervals for which $F$ is increasing | 1 |

**Question 18 (d)** **(2 marks)**

| Solution | |
|---|---|
| Where points of inflection occur $F''(x) = f'(x) = 0$. Hence $x \approx -0.8$ and $-3.2$. | |
| **Mathematical behaviours** | **Marks** |
| • states $F''(x) = f'(x) = 0$. | 1 |
| • states the approximate $x$ value ($\pm 0.1$) of both stationary points | 1 |

**Question 19** **(4 marks)**

Solution

$e^x = e^{-x+4}$
$x = -x + 4$
$x = 2$

$e^x = e$
$x = 1$

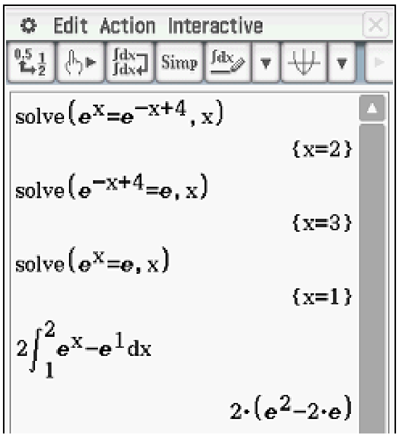
$e^{-x+4} = e$
$x = 3$

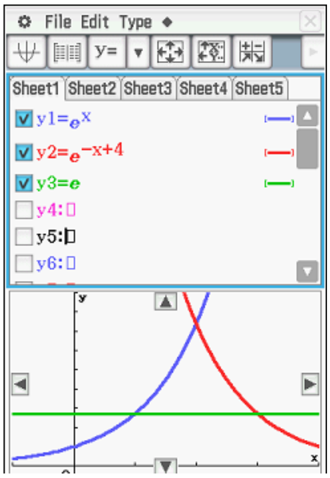
$$A = 2\int_1^2 e^x - e \; dx$$

$$= 2\left[e^x - ex\right]_1^2$$

$$= 2\left[\left(e^2 - 2e\right) - \left(e^1 - e\right)\right]$$

$$= 2(e^2 - 2e)$$




| Mathematical behaviours | Marks |
|---|---|
| • determines one $x$ co-ordinate of intersections | 1 |
| • determines all $x$ co-ordinates of intersections | 1 |
| • states appropriate integral to determine area | 1 |
| • determines exact area | 1 |

**Question 20 (a)** **(2 marks)**

| Solution | |
|---|---|
| $a(t) = 6t + 4$<br>$v(t) = 3t^2 + 4t + c$<br>$t = 0, v = 0 \Rightarrow c = o$<br>$\therefore v(t) = 3t^2 + 4t$ | $\int 6x+4dx$<br>$3\cdot x^2+4\cdot x$ |

| Mathematical behaviours | Marks |
|---|---|
| • anti-differentiates correctly<br>• uses initial conditions to establish $c$ = 0 | 1<br>1 |

**Question 20 (b)** **(2 marks)**

| Solution | |
|---|---|
| If the particle changes direction, $v(t) = 0$<br>$ie\ \ 3t^2 + 4t = 0$<br>$ie\ \ t(3t+4) = 0$<br>$ie\ \ t = 0, \frac{-4}{3}$ Hence it does not change direction | $\text{solve}(3\cdot x^2+4\cdot x=0, x)$<br>$\left\{x=0, x=-\frac{4}{3}\right\}$ |

| Mathematical behaviours | Marks |
|---|---|
| • equates $v(t) = 0$<br>• solves equation and states that particle does not change direction | 1<br>1 |

**Question 20 (c)** **(2 marks)**

| Solution | |
|---|---|
| Total distance travelled $= \int_0^3 3t^2 + 4t \quad dt = 45$<br>Hence average speed = 15$ms^{-1}$ | $\int_0^3 3\cdot x^2+4\cdot x dx$<br>$45$ |

| Mathematical behaviours | Marks |
|---|---|
| • states integral required to determine total distance travelled<br>• determines average speed | 1<br>1 |

**Question 20 (d)** **(2 marks)**

| Solution | |
|---|---|
| $v(2) = 20ms^{-1},\ a(2) = 16ms^{-2}$<br>Hence the particle is moving with a positive velocity and is gaining speed | $3\cdot x^2+4\cdot x \mid x=2$<br>$20$<br>$6x+4 \mid x=2$<br>$16$ |

| Marking key/mathematical behaviours | Marks |
|---|---|
| • evaluates at least one of $v(2)$ and $a(2)$<br>• states the particle is moving with a positive velocity/to the right and is speeding up | 1<br>1 |

**Question 21 (a)** **(1 mark)**

| Solution | |
|---|---|
| $S = 2\pi rh + 2\pi r^2$<br>$= 2\pi r 6r + 2\pi r^2$<br>$= 14\pi r^2$ | |
| Mathematical behaviours | Marks |
| • determines expression | 1 |

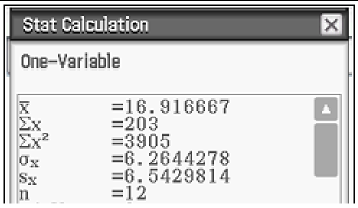
**Question 21 (b)** **(4 marks)**

| Solution | |
|---|---|
| $\frac{dS}{dr} = 28\pi r$ $\qquad \frac{\delta r}{r} = 0.048$<br>$\delta S \approx \frac{dS}{dr} \times \delta r$<br>$\delta S \approx 28\pi r \times \delta r$<br>$\delta S \approx 28\pi r \times 0.048r$<br>$r = \frac{6.541}{2}$<br>$\therefore \delta S \approx 28\pi (0.048)(\frac{6.541}{2})^2$<br>$\approx 45.2$<br>$\therefore$ Approximately 45.2 cm$^2$ | |
| Mathematical behaviours | Marks |
| • differentiates expression | 1 |
| • uses $\delta r = 0.048r$ and $r = \frac{6.541}{2}$ | 1 |
| • obtains expression for $\delta S$ | 1 |
| • determines approximate increase in metal required including unit | 1 |

**Question 22 (a)** **(2 marks)**

| Solution | |
|---|---|
| In the northern hemisphere highest temperatures occur in the middle of the year, whereas in the southern hemisphere highest temperatures occur at the beginning and end of the year. Since the data show high temperatures in the middle of the year, the city is more likely to be in the northern hemisphere. | |
| Mathematical behaviours | Marks |
| • states more likely hemisphere | 1 |
| • valid reasoning | 1 |

**Question 22 (b)** **(2 marks)**

| Solution | |
|---|---|
| The average maximum temperature values are $8, 8, 10, 13, 18, 22, 25, 25, 24, 21, 17$ and $12$.<br>Mean = $16.92$<br>So the estimated average maximum temperature is $16.92$℃ |  |



| Mathematical behaviours | Marks |
|---|---|
| • states an appropriate calculation to determine the mean | 1 |
| • determines the mean | 1 |

**Question 22 (c)** **(2 marks)**

| Solution |
|---|
| The estimated standard deviation of the temperatures is $6.26$℃. |

| Mathematical behaviours | Marks |
|---|---|
| • determines the standard deviation | 1 |
| • states standard deviation to at least 1 decimal place | 1 |

**Question 22 (d)** **(2 marks)**

| Solution |
|---|
| The estimated average maximum temperature is $16.92 \times 1.8 + 32 \approx 62.46°\mathrm{F}$<br>The estimated standard deviation is $6.26 \times 1.8 \approx 11.27°\mathrm{F}$ |

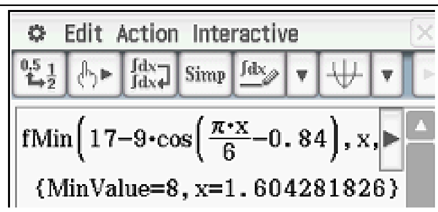
| Mathematical behaviours | Marks |
|---|---|
| • states average in $°F$ | 1 |
| • states standard deviation in $°F$ | 1 |

**Question 22 (e)** **(2 marks)**

| Solution |
|---|
| In the model $y = A - B\cos\left(\frac{\pi t}{6} - 0.84\right)$ the average value is $A$, and the values range from $A - B$ to $A + B$. So $A \approx 17$ and $B \approx 9$ |

| Mathematical behaviours | Marks |
|---|---|
| • determines $A$ | 1 |
| • determines $B$ | 1 |

**Question 22 (f)** **(2 marks)**

| Solution | |
|---|---|
| $y = A - B\cos\left(\frac{\pi t}{6} - 0.84\right)$ has a minimum when $\frac{\pi t}{6} - 0.84 = 0$, i.e. $t \approx 1.6$<br>The nearest integer value is $2$. So according to the model, the maximum daily temperatures are least when $t = 2$, i.e. in February. |  |



| Mathematical behaviours | Marks |
|---|---|
| • obtains $t \approx 1.6$ | 1 |
| • states the lowest maximum is reached in February | 1 |